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Measuring Dependence in Uncertainty Should Start in the Introduction to Probability and Statistics

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Measuring Dependence in Uncertainty Should Start in the Introduction to Probability and Statistics

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Outline

- **Some previous results on measuring dependence**
- **General observations and rules**
- **Illustrations**
 - **Non-numeric variables**
 - **Numeric Variables -Dependence in Bivariate Distributions**
- **More Illustrations**
 - **Dependence in Politics**
 - **Local dependence in reliability systems**

Introduction

- ▶ In several publications we noticed an idea how probability tools can be used to measure strength of dependence between random events
- ▶ In the present talk we propose to use it for measuring magnitude of local dependences between random variables.
- ▶ As illustration, we demonstrate how it works as a measuring tool in the complicated world of politics and in reliability models.
- ▶ Short illustration is discussed on the use of these measures in already known previously popular results for non-numeric uncertain variables.

How to INDICATE dependence?

The dependence in the world of the uncertainty is a complex concept.

- ▶ **Textbooks avoid discussions in this regard.**
- ▶ **In the classical approach conditional probability is used to determine if two events are dependent, or not: A and B are independent when the probability for their joint occurrence equals to the product of the probabilities for their individual appearance, i.e. when**

$$P(A \cap B) = P(A).P(B)$$

- ▶ **Otherwise, the two events are dependent.**

How to measure dependence?

- ▶ To measure dependence between random events

B. Dimitrov (2010, Some Obreshkov Measures of Dependence and Their Use, *Compte Rendus de l'Acad. Bulgare des Sci.*, v. 63, No.1, pp. 15-18)

revived some measures of dependence for random events based on notion of probabilities of the events.

From that discussion and among the four proposed measures **we selected the Regression coefficients as suitable measure of magnitude of dependence** when the two events are dependent.

Regression Coefficients as Measures of dependence between random events

- **Definition 1.** *Regression coefficient* of the event **A** with respect to the event **B** is called the difference between the conditional probability for the event **A** given the event **B**, and the conditional probability for the event **A** given the complementary event \overline{B} , namely

$$R_B(A) = P(A | B) - P(A | \overline{B})$$

- This *measure of the dependence of the event A on the event B*, is *directed dependence*.
- The regression coefficient is always defined, for any pair of events **A** and **B** (zero, sure, arbitrary).
- The regression coefficient of **B with respect to A** is defined *symmetrically*

Properties of Regression coefficients

- **(r1)** The equality to zero $R_B(A) = R_A(B) = 0$ holds only if the events **A** and **B** are independent.
- **(r2)** $R_A(A) = 1$; $R_{\bar{A}}(A) = -1$.
- **(r3)** $R_B(\sum A_j) = \sum R_B(A_j)$
- **(r4)** $R_S(A) = R_{\emptyset}(A) = 0$
- **(r5)** The regression coefficients are numbers with equal signs
- To be valid $R_B(A) = R_A(B)$ it is necessary and sufficient to have

$$P(A)[1 - P(A)] = P(B)[1 - P(B)]$$

Regression coefficients as measure of dependence between random events.

- **The relations**

$$R_B(A) = \frac{P(A \cap B) - P(A)P(B)}{P(B)[1 - P(B)]}$$

and

$$R_A(B) = \frac{P(A \cap B) - P(A)P(B)}{P(A)[1 - P(A)]}$$

explain when it will be $R_B(A) = R_A(B)$.

These properties, and next, may be used as exercises in the classroom.

Regression coefficients - properties

(r6) The regression coefficients $R_B(A)$ and $R_A(B)$ are numbers between -1 and 1 , i.e. they satisfy the inequalities

$$-1 \leq R_B(A) \leq 1 \quad -1 \leq R_A(B) \leq 1$$

(r6.1) The equality $R_B(A) = 1$ holds only when **A coincides** (is equivalent) **with the event B**.

Then is also valid the equality $R_A(B) = 1$;

(r6.2) The equality $R_B(A) = -1$ holds only when event **A coincides** (or is equivalent) with the event \overline{B} - the **complement of B**.

Then is also valid $R_A(B) = -1$, and respectively $\overline{\overline{A}} = B$.

Regression coefficients – a proposition

- ❖ **In our opinion, it is possible one event to have stronger dependence on the other than the reverse.**
- ❖ **This measure suits for measuring the magnitude of dependence between events.**
- ❖ **The distance of the regression coefficient from the zero (where the independence is) could be used to classify the strength of dependence, e.g. (taken from some textbooks)**
 - **almost independent** (when $R_A(B) < .05$) ;
 - **weakly dependent** (when $.05 < |R_A(B)| < .2$) ;
 - **moderately dependent** (when $.2 < |R_A(B)| < .45$) ;
 - **in average dependent** (when $.45 < |R_A(B)| < .8$) ;
 - **strongly dependent** (when $|R_A(B)| > .8$) ;

4. Correlation between two random events

- **Definition 3.** Correlation coefficient between two events A and B is defined by the number

$$R_{A,B} = \pm \sqrt{r_B(A) \cdot r_A(B)}$$

Its sign, plus or minus, is the sign of either of the two regression coefficients.

- An equivalent representation

$$R_{A,B} = \frac{P(A \cap B) - P(A)P(B)}{\sqrt{P(A)P(\bar{A})} \sqrt{P(B)P(\bar{B})}}$$

Correlation (properties)

- **$\rho 1$.** It is fulfilled $R_{A,B} = 0$ if and only if the two events A and B are independent.
- **$\rho 2$.** It is fulfilled $-1 \leq R_{A,B} \leq 1$.
- **$\rho 2.1$.** The equality to 1 holds if and only if the events A and B are equivalent, i.e. when $A = B$.
- **$\rho 2.2$.** The equality $R_{A,B} = -1$ holds if and only if the events A and \overline{B} are equivalent

4. Correlation Properties (continued)

- **ρ_3** . The correlation coefficient has the same sign as the other measures of the dependence between two random events A and B , and this is the sign of the connection.
- **ρ_4** . The knowledge of $R_{A,B}$ allows calculating the posterior probability of one of the events under the condition that the other one is occurred. For instance, $P(B | A)$ will be determined by the rule

$$P(B | A) = P(B) + R_{A,B} \sqrt{\frac{P(\bar{A})P(B)P(\bar{B})}{P(A)}}$$

- The net increase, or decrease in the posterior probability compare to the prior probability equals to the quantity added to $P(B)$, and depends only on the value of the mutual correlation.

4. Correlation (continued)

- $P(B | \bar{A}) = P(B) - R_{A,B} \sqrt{\frac{P(A)P(B)P(\bar{B})}{P(\bar{A})}}$
- **$\rho 5$.** It is fulfilled $R_{\bar{A},B} = R_{A,\bar{B}} = -R_{A,B}$; $R_{\bar{A},\bar{B}} = R_{A,B}$
- **$\rho 6$.** $R_{A,A} = 1$; $R_{A,\bar{A}} = -1$; $R_{A,S} = R_{A,\emptyset} = 0$
- **$\rho 7$. Particular Cases.** When $A \subset B$, then

$$R_{A,B} = \sqrt{\frac{P(A)P(\bar{B})}{P(\bar{A})P(B)}}; \text{ If } A \cap B = \emptyset, \text{ then } R_{A,B} = -\sqrt{\frac{P(A)P(B)}{P(\bar{A})P(\bar{B})}}$$

4. Correlation (continued)

- The use of the numerical values of the correlation coefficient is similar to the use of the two regression coefficients.
- As closer is $R_{A,B}$ to the zero, as “closer” are the two events A and B to the independence.
- Let us note once again that $R_{A,B} = 0$ if and only if the two events are independent.

4. Correlation (continued)

- As closer is $R_{A,B}$ to 1, as “dense one within the other” are the events A and B , and when $R_{A,B} = 1$, the two events coincide (are equivalent).
- As closer is $\frac{R_{A,B}}{B}$ to -1, as “dense one within the other” are the events A and \overline{B} , and when $R_{A,B} = -1$ the two events coincide (are equivalent).
- These interpretations seem convenient when conducting research and investigations associated with qualitative (non-numeric) factors and characteristics.
- Such studies are common in sociology, ecology, jurisdictions, medicine, criminology, design of experiments, and other similar areas.

4. Correlation (continued)

- Freshe-Hoeffding inequalities for the Correlation Coefficient

$$\max \left\{ -\sqrt{\frac{P(A)P(B)}{P(\bar{A})P(\bar{B})}}, -\sqrt{\frac{P(\bar{A})P(\bar{B})}{P(A)P(B)}} \right\} \leq R(A, B) \leq \min \left\{ \sqrt{\frac{P(A)P(\bar{B})}{P(\bar{A})P(B)}}, \sqrt{\frac{P(\bar{A})P(B)}{P(A)P(\bar{B})}} \right\}$$

4. Correlation (continued)

- *Example 1 (continued)*: We have the numerical values of the two regression coefficients and from the previous section. In this way we get

$$R_{A,B} = \sqrt{(.3368)(.2174)} = .2706.$$

- Analogously to the use of the regression coefficients, the numeric value of the correlation coefficient could be used for classifications of the degree of the mutual dependence.
- **The correlation coefficient is a number in-between the two regression coefficients. It is symmetric and absorbs the misbalance (the asymmetry) in the two regression coefficients. It is a balanced measure of dependence between the two events.**

5. Empirical estimations

- The measures of dependence between random events are made of their probabilities. It makes them very attractive and in the same time **easy for statistical estimation and practical use.**

5. Empirical Estimations (contd)

- Let in N independent experiments (observations) the random event A occurs k_A times, the random event B occurs k_B times, and the event $A \cap B$ occurs $k_{A \cap B}$ times. Then statistical estimators of our measures of dependence will be respectively:

$$\hat{\delta}(A, B) = \frac{k_{A \cap B}}{N} - \frac{k_A}{N} \cdot \frac{k_B}{N}$$

5. Empirical Estimations (contd)

- The estimators of the two regression coefficients are

$$\hat{r}_A(B) = \frac{\frac{k_{A \cap B}}{N} - \frac{k_A}{N} \cdot \frac{k_B}{N}}{\frac{k_A}{N} \left(1 - \frac{k_A}{N}\right)}; \quad \hat{r}_B(A) = \frac{\frac{k_{A \cap B}}{N} - \frac{k_A}{N} \cdot \frac{k_B}{N}}{\frac{k_B}{N} \left(1 - \frac{k_B}{N}\right)}$$

- The correlation coefficient has estimator

$$\hat{R}(A, B) = \frac{\frac{k_{A \cap B}}{N} - \frac{k_A}{N} \cdot \frac{k_B}{N}}{\sqrt{\frac{k_A}{N} \left(1 - \frac{k_A}{N}\right) \frac{k_B}{N} \left(1 - \frac{k_B}{N}\right)}}$$

5. Empirical Estimations (contd)

- **These estimators may be simplified** when the numerator and denominator are multiplied by appropriate quantity. We not go into detail.
- **The estimators are all consistent**; the estimator of the connection $\delta(A,B)$ is also unbiased, i.e. there is no systematic error in this estimate.
- The estimators can be used in practice with reasonable interpretations and explanations

6. Some warnings and some recommendations

- **The measures of dependence between random events are not transitive.**
- It is possible A to be positively associated with B , event B to be positively associated with C , but the A to be negatively associated with C .
- Example: A and B compatible (non-empty intersection); B and C compatible, A and C - mutually exclusive, and with a negative connection.
- For non-exclusive pairs (A, B) and (B, C) every kind of dependence is possible.
- **More precisions at this point deserve attention.**

6. *Some recommendations (contd)*

- One can use the measures of dependence to compare degrees of dependence.
- We recommend the use of Regression Coefficient for measuring degrees of dependence.
- For instance, let $|r_B(A)| \leq |r_C(A)|$

then we say that the event A has stronger association with C compare to its association with B .

- In this way some ranks of associations of a given event can be established for any collection of other events.

From Events to Random Variables

- The introduced measures allow **to see the interaction between** any pair of numeric **r.v.'s (X, Y)** throughout the **sample space**
- **Understand and use the local dependence.**
- **Let $F(x, y) = P(X \leq x, Y \leq y)$ - the joint c.d.f.**
- **Marginals $F(x) = P(X \leq x)$, $G(y) = P(Y \leq y)$.**

From Events to Random Variables

- Introduce the events

- $A = \{x \leq X \leq x + \Delta_1 x\}; \quad B = \{y \leq Y \leq y + \Delta_2 y\},$

for any $x, y \in (-\infty, \infty)$.

- *Then the measures of dependence between events A and B turn into a measure of local dependence between the pair of r.v.'s X and Y on the rectangle*

$$D = [x, x + \Delta_1 x] \times [y, y + \Delta_2 y].$$

From Events to Random Variables

- Naturally, they can be named and calculated as follows:
- **Regression coefficient** of X with respect to Y , and of Y with respect to X on the rectangle $D = [x, x + \Delta_1 x] \times [y, y + \Delta_2 y]$. By Definition 1 we get

$R_Y((X, Y) \text{ on } D) =$

$$\frac{\Delta_D F(x, y) - [F(x + \Delta_1 x) - F(x)][G(y + \Delta_2 y) - G(y)]}{[F(x + \Delta_1 x) - F(x)]\{1 - [F(x + \Delta_1 x) - F(x)]\}}$$

- Here by $\Delta_D F(x, y)$ is denoted the **two dimensional finite difference** for the function $F(x, y)$ on rectangle $D = [x, x + \Delta_1 x] \times [y, y + \Delta_2 y]$.

From Events to Random Variables

- Namely

$$\Delta_D F(x,y) = F(x+\Delta_1 x, y+\Delta_2 y) - F(x+\Delta_1 x, y) - F(x, y+\Delta_2 y) + F(x, y).$$

- In an analogous way is defined $R_x((X,Y) \text{ on } D)$. Just denominator in the expression is changed respectively.
- **Correlation coefficient** $R_y((X,Y) \text{ on } D)$ between the r.v.'s X and Y on rectangle $D=[x, x+\Delta_1 x] \times [y, y+\Delta_2 y]$ can be presented in similar way by the **use of Definition 2**. We omit detailed expressions as something obvious.

From Events to Random Variables

- The local dependence at a value $(X=i, Y=j)$ for a pair of discrete distributed r.v. (X, Y) .
- **Regression coefficient** of X with respect to Y , and of Y with respect to X at a value $(X=i, Y=j)$ is determined by the rule

- $R_Y(X=i, Y=j) =$

$$\frac{P(X=i, Y=j) - P(X=i)P(Y=j)}{P(X=i)[1 - P(X=i)]}$$

- The local **correlation coefficient** values of the two r.v.'s

$$R_{X,Y}(X=i, Y=j) = \frac{p(i, j) - p_{i.} p_{.j}}{\sqrt{p_{i.} (1 - p_{i.})} \sqrt{p_{.j} (1 - p_{.j})}}$$

Categorical variables

As another illustration of the measures of dependence we analyze an example from Alan Agresti *Categorical Data Analysis, 2006*. The table represents data about the yearly income of people and the job satisfaction.

Job Satisfaction

Income US \$\$	Very Dissatisf	Little Satisfied	Moderately Satisfied	Very Satisfied	Total Marginally
< 6,000	20	24	80	82	206
6,000– 15,000	22	38	104	125	289
15,000- 25,000	13	28	81	113	235
> 25,000	7	18	54	92	171
Total	62	108	319	412	901

Categorical variables (continued)

Table 2: Joint and marginal probability distributions (Income, Job Satisfaction) $P_{i,j}, P_{i.}, P_{.j}$

Job Satisfaction					
Income US \$\$	Very Dissatisfied	Little Satisfied	Moderately Satisfied	Very Satisfied	Total (marginal) distribution
< 6,000	.02220	.02664	.08879	.09101	.22864
6,000–15,000	.02442	.04217	.11543	.13873	.32075
15,000-25,000	.01443	.03108	.08990	.12542	.26083
> 25,000	.00776	.01998	.05993	.10211	.18978
Total (marginal) distribution	.06881	.11987	.35405	.45727	1.00000

Categorical Variables (Regr. Coeff. 1)

Tab 3: Empirical regression coefficient between particular levels of income and job satisfaction $r_{Satisfaction_j}(IncomeGroup_i)$

Job Satisfaction				
Income US \$\$	Very Dissatisfied	Little Satisfied	Moderately Satisfied	Very Satisfied
< 6,000	0.100932704	-0.00727	0.034281	-0.05456
6,000–15,000	0.036663063	0.035276	0.00817	-0.03199
15,000-25,000	-0.05489976	-0.00176	-0.0107	0.024782
> 25,000	-0.08269601	-0.02625	-0.03175	0.061768

Categorical Variables (Regr. Coeff. 2)

Table 5: Empirical regression coefficients between particular levels of job satisfaction and income $r_{Income}(\text{Job Satisfaction})$

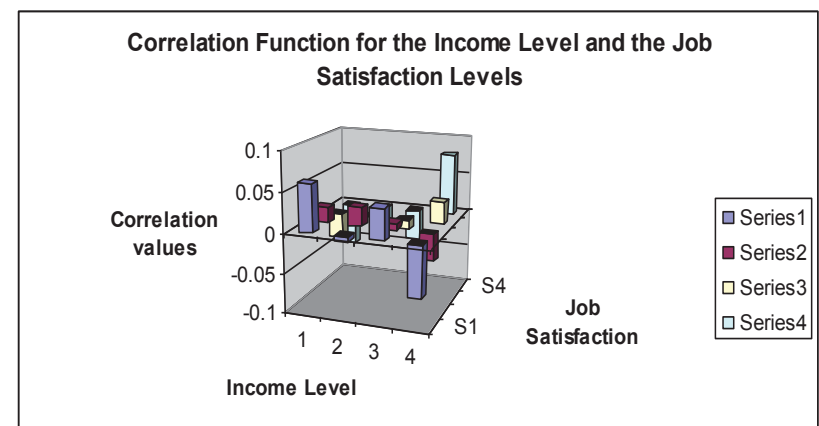
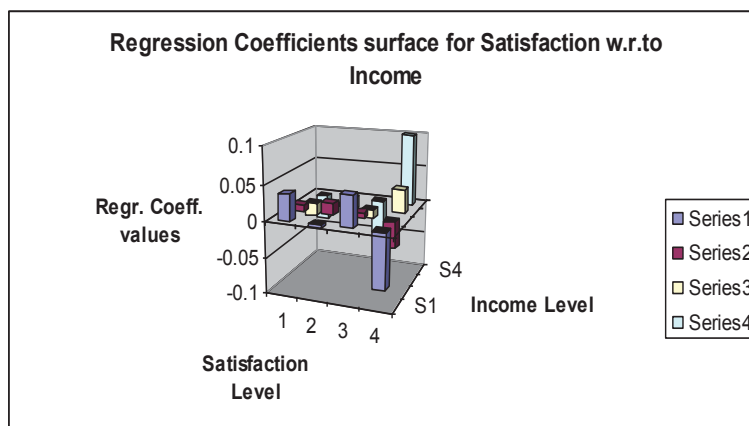
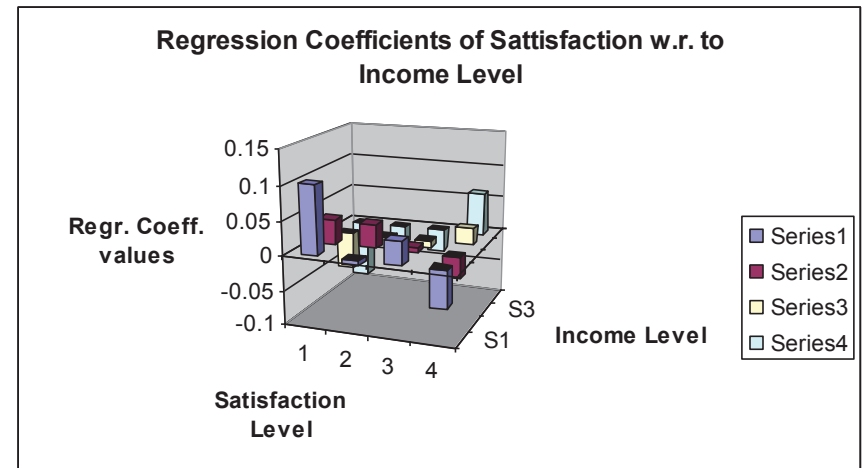
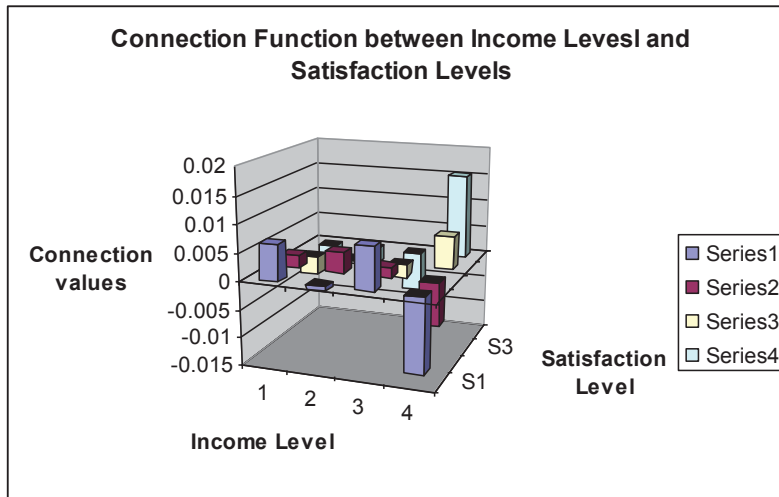
Income US \$\$	Job Satisfaction			
	Very Dissatisfied	Little Satisfied	Moderately Satisfied	Very Satisfied
< 6,000	0.03667013	-0.00435	0.044454	-0.07677
6,000–15,000	0.01078257	0.017082	0.008576	-0.03644
15,000-25,000	-0.01824561	-0.00096	-0.01269	0.0319
> 25,000	-0.03446045	-0.01801	-0.04723	0.099694

Categorical Variables (Correl. Coeff.)

Tab 6: Empirical correlation coefficient between particular income and job satisfaction levels
R(Income_i, Satisfaction_j)

Job Satisfaction				
Income US \$\$	Very Dissatisfied	Little Satisfied	Moderately Satisfied	Very Satisfied
< 6,000	0.060838	- 0.005623	0.039037	- 0.064721
6,000–15,000	0.019883	0.024548	0.008371	- 0.034144
15,000-25,000	- 0.031649	- 0.001302	- 0.011653	0.028117
> 25,000	- 0.053383	- 0.02174	- 0.038723	0.078472

Categorical variables - Graphic comparison



Local dependence structure: The simplest Bivariate Poisson distribution with dependent components

- The bivariate discrete distribution presented by the three positive parameters (λ, μ, ν) family

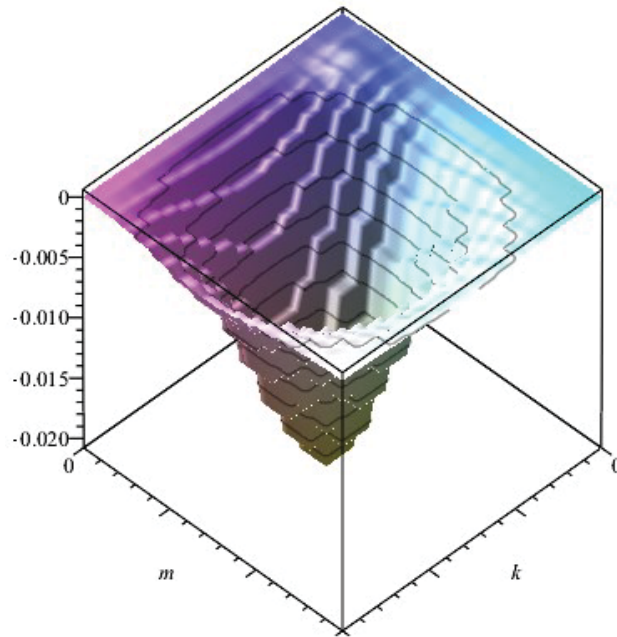
$$P(X = x, Y = y) = e^{-\lambda-\mu-\nu} \frac{\lambda^x}{x!} \cdot \frac{\mu^y}{y!} \sum_{k=0}^{\min(x,y)} \binom{x}{k} \binom{y}{k} k! \left(\frac{\nu}{\lambda\mu}\right)^k$$

- Here $x, y = 0, 1, 2, \dots$ are the possible values of the variables X and Y . If $M_1, M_2,$ and M_3 are three independent Poisson distributed r.v.'s with parameters $\lambda, \mu,$ and ν correspondingly, then dependence between X and Y comes from the fact that X is distributed as the sum $X = M_1 + M_3,$ and $Y = M_2 + M_3.$ Inclusion of M_3 in both sums makes them dependent.
- The marginal distributions of X and Y are Poisson with parameters $\lambda + \mu,$ and $\lambda + \nu$ respectively

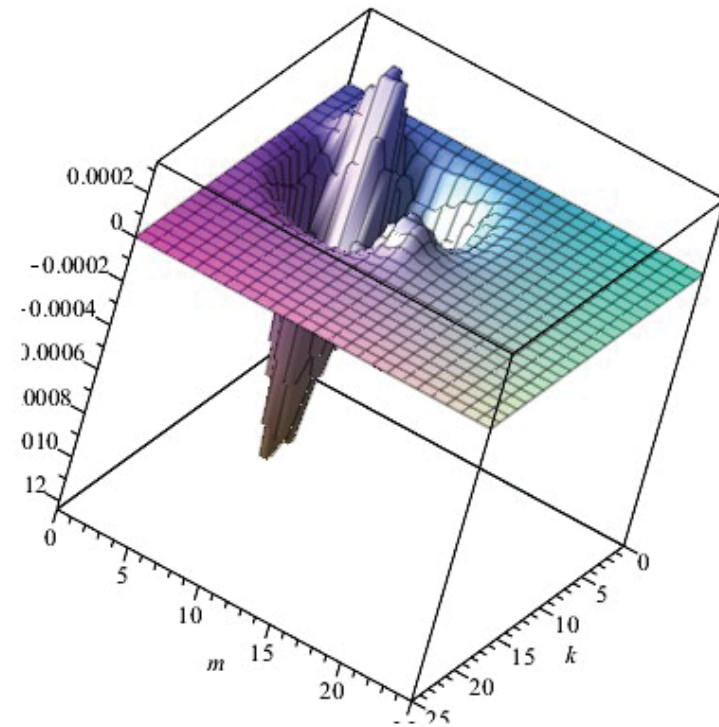
The Bivariate Poisson distribution with dependent components

- We avoid explicit cumbersome expressions for $\delta_{x,y}(x,y)$, the two regression coefficient functions $R_x(Y; x,y)$ and $R_y(Y; x,y)$, and the correlation function $\rho_{x,y}(x,y)$
- Local Dependence at each point (x,y) with integer coordinates is programmed for the values **$\lambda=3$, $\mu=2$ and $\nu=5$**
- The graphs of these functions are shown on next graphs.
- As the ancient Greek geometers use to say, **just watch and conclude at what point what kind of dependence works, and what is its strength.**

The Bivariate Poisson distribution with dependent components

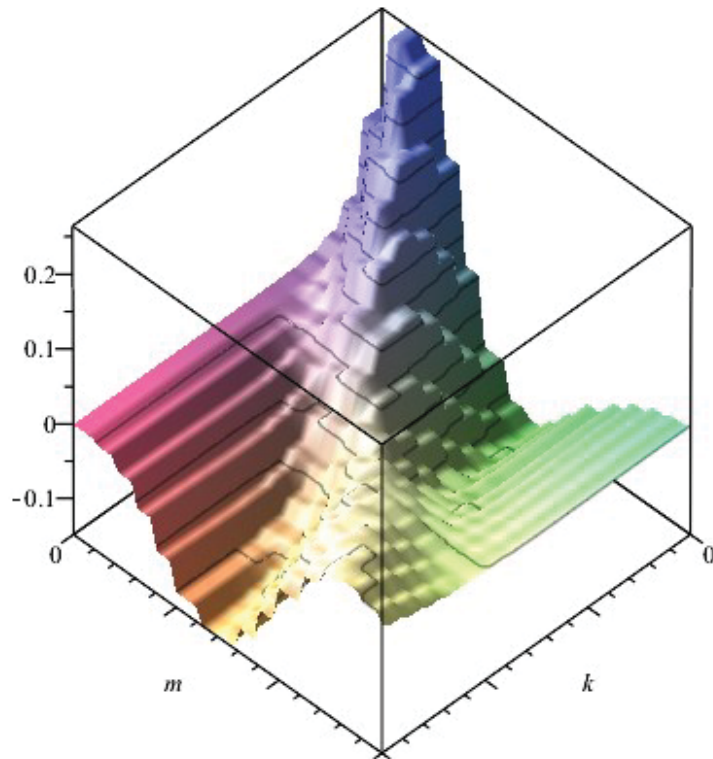


Connection function $\delta_{X,Y}(x,y)$

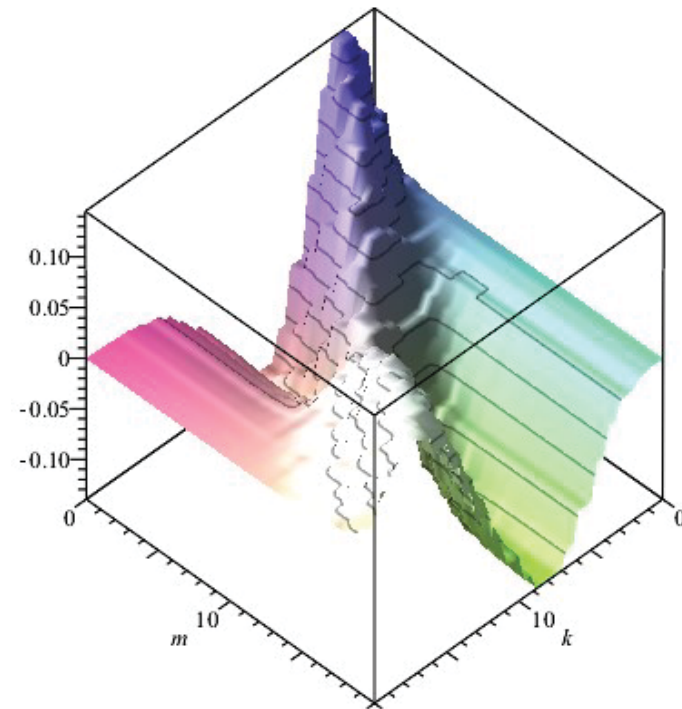


The correlation function $\rho_{X,Y}(x,y)$

The Bivariate Poisson distribution with dependent components



The regression coeff. F-n $R_Y(X; x, y)$



The regression coeff. F-n $R_X(Y; x, y)$

Local dependence structure in the political charts

- Esa and Dimitrov (2013a) have shown that a **multinomial model** describes the spectrum of the party's life in a country. **With N independent active *free individuals***, the coordinates of **the random vector** (X_0, X_1, \dots, X_r) represent the number of individuals members of each party. They are distributed by the multinomial law

- $P(X_0=k_0, X_1=k_1, \dots, X_r=k_r) =$
$$\frac{N!}{k_0! k_1! \dots k_r!} P_0^{k_0} P_1^{k_1} \dots P_r^{k_r}$$

$$k_0 + k_1 + \dots + k_r = N$$

Local dependence structure in the political charts

- The regression coefficients and correlation coefficients of the local dependence between any two components of the political life in the country are obtained from

$$P(X_i=n, X_j=m) = \frac{N!}{n!m!(N-n-m)!} P_i^n P_j^m (1-P_i-P_j)^{N-n-m}$$

$$P(X_i = n) = \frac{N!}{n!(N-n)!} P_i^n (1-P_i)^{N-n}$$

and

$$P(X_j = m) = \frac{N!}{m!(N-m)!} P_j^m (1-P_j)^{N-m}$$

Local dependence structure in the political charts –An Example

- ▶ Let us assume that in the main model we have $r=4$; $\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$; $\mu_1=2$, $\mu_2=3$, $\mu_3=4$, and $\mu_4=5$. Then

$$P_0 = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)^{-1} = \frac{60}{137}$$

- ▶ Respectively $P_1=30/137$; $P_2=20/137$; $P_3=15/137$

and $P_4=12/137$.

Assuming for simplicity $N=1000$, we draw a **surface of the local Regression coefficient of the dominating party 1 with respect to the next leading party 2 (Fig. 1.)**. It is shown on the next figure.

Next to it is the regression coefficient measure of dependence of the weakest party 4 on the strongest party 1 (Fig. 2.).

Local dependence structure in the political charts –An Example

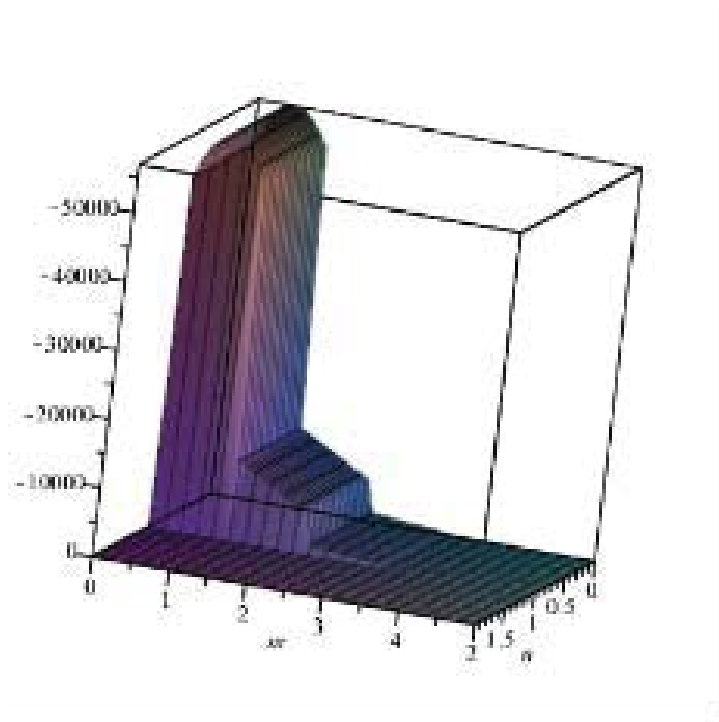


Fig. 1. Local Regression coefficient surface dominating party 1 with respect to next leading party 2.

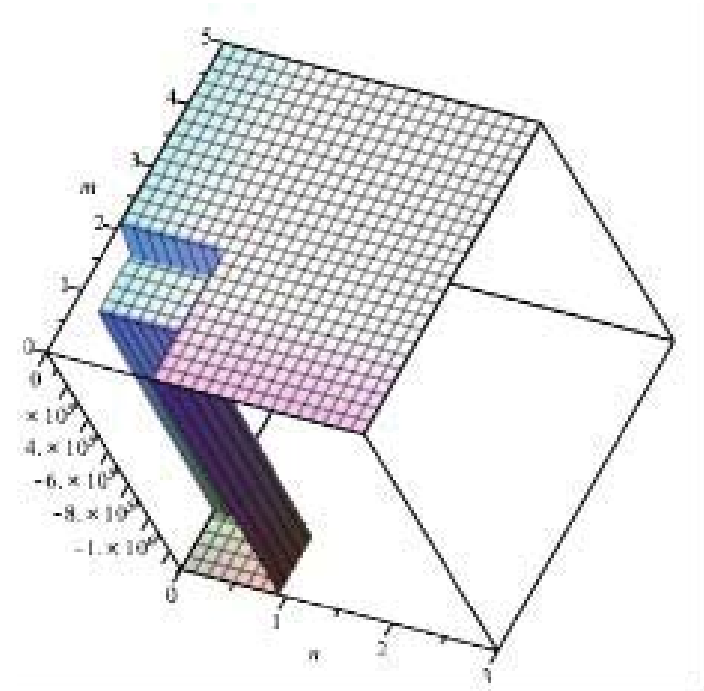


Fig. 2. Regression coefficient surface - the weakest party 4 with respect to strongest party 1

Local dependence structure in the political charts –An Example - discussion

- We see that **the local dependences between** parties (the strongest to next leading) **are negative when both parties have low results in votes. Dependence is negligible when votes are higher.**
- Dependence of the **weakest party vs. strongest** one goes low flat when it gets low number of votes.

Local dependence in reliability systems

- ▶ We focus on **two traditional systems** of independent components,
 - System in series** and
 - System in parallel.**
- ▶ **1. We study the regression coefficients of a component with respect to the system, and**
- ▶ **2. Regression coefficient of the system with respect to a component**
 - How these measures of dependence change in time during the work of the system.**
- ▶ **For simplicity consider system of just two components.**
- ▶ **REASON:** considering one component, everything else can be aggregated as a second component.

System in series.

Assume, components have **exponentially distributed live times** with parameters λ_1 and λ_2 .

Then the **reliability function** at time t (**event B**) equals

$r(t) = e^{-(\lambda_1 + \lambda_2)t}$ The probability that component 1 functions (**event A**) is $e^{-\lambda_1 t}$.

The **regression coefficient of the system with respect to component 1** is

$$R_1(S) = \frac{r(t) - r(t)e^{-\lambda_1 t}}{e^{-\lambda_1 t} (1 - e^{-\lambda_1 t})} = e^{-\lambda_2 t}$$

System in series

- The **regression coefficient of the component 1 with respect to the system at time t** is given by

$$R_S(1) = \frac{r(t) - r(t)e^{-\lambda_1 t}}{r(t)[1 - r(t)]} = \frac{1 - e^{-\lambda_1 t}}{1 - e^{-(\lambda_1 + \lambda_2)t}}$$

- The **correlation coefficient between system reliability and the component** reliability are changing during the time according to

$$\rho_{s,1}(t) = \sqrt{\frac{e^{-\lambda_2 t} (1 - e^{-\lambda_1 t})}{1 - e^{-(\lambda_1 + \lambda_2)t}}} \quad \rho_{s,2}(t) = \sqrt{\frac{e^{-\lambda_1 t} (1 - e^{-\lambda_2 t})}{1 - e^{-(\lambda_1 + \lambda_2)t}}}$$

System in series

Notice that all dependences are positive. Graphs of these functions of local dependence in time for $\lambda_1=1$ and $\lambda_2=2$ are shown on next figures (Fig. 3 and Fig.4)

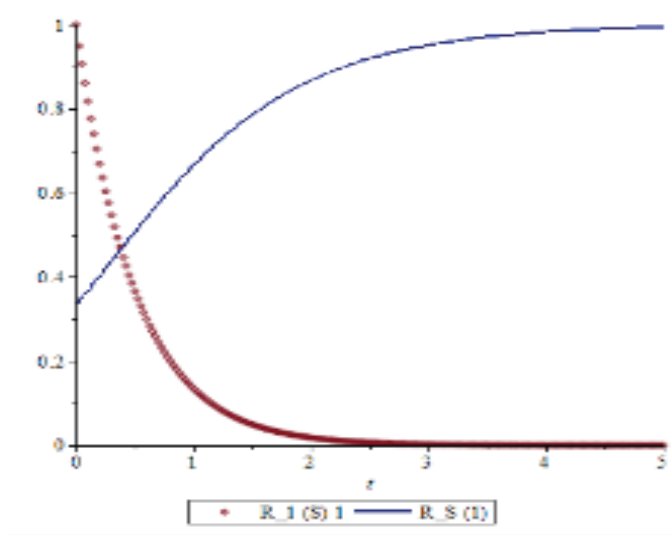


Fig. 3. Regression coeff. between between system reliability and the strongest component ($\lambda=1$) **(time dependence)**

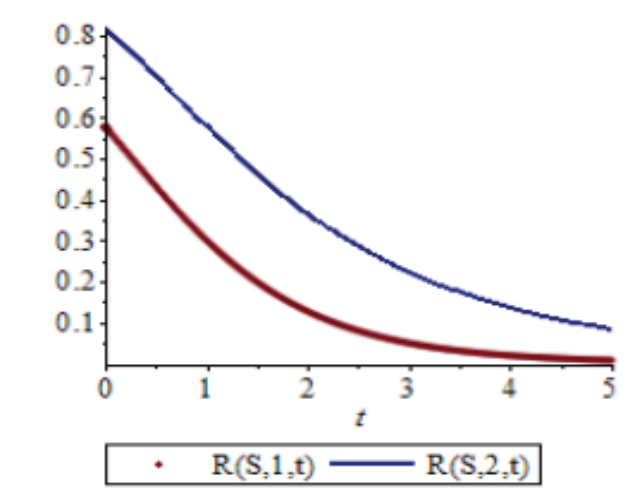


Fig. 4. Regression coefficients between system reliability and weakest component ($\lambda=2$) **(time dependence)**

System in series (discussion)

The Regr. coeff. for the weakest component 2 w.r. to system and system w.r. to component , decrease when the time increases, and behave similarly;

The regression coefficients between the system and the strongest component behave different:

Local dependence **$R_1(S)$ approaches 0** with the time (**system becomes independent on component 1 with the time increase**);

The local dependence $R_S(1)$ of strongest component 1 on the system reliability approaches 1 with the time increase (Fig.3).

System in parallel

Assume again **both components lives exponential** with parameters λ_1 and λ_2 .

The **reliability function** at time t (event B) equals

$$r(t) = 1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$

- The probability that component 1 functions (event A)

is $e^{-\lambda_1 t}$

- The **regression coefficient** of the **system with respect to component 1** is

$$R_1(S) = \frac{1 - r(t)}{1 - e^{-\lambda_1 t}} = 1 - e^{-\lambda_2 t}$$

System in parallel

- The regression coefficient of the component 1 with respect to the system at time t is given by

$$R_S(1) = \frac{e^{-\lambda_1 t} - r(t)e^{-\lambda_1 t}}{r(t)[1 - r(t)]} = \frac{e^{-\lambda_1 t}}{1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}$$

- The correlation coefficient between system reliability and the component reliability in time is

$$\rho_{S,1}(t) = \sqrt{\frac{e^{-\lambda_1 t} (1 - e^{-\lambda_2 t})}{1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}} \quad ; \quad \rho_{S,2}(t) = \sqrt{\frac{e^{-\lambda_2 t} (1 - e^{-\lambda_1 t})}{1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}}$$

System in parallel – an example

- ▶ **All dependences are positive.** Graphs of these functions of local dependence in time for $\lambda_1=1$ and $\lambda_2=2$ are shown on Fig. 5 and Fig .6.

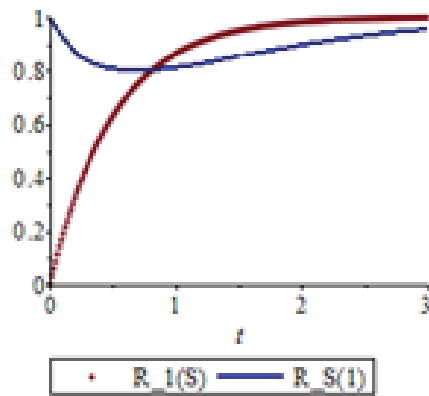


Fig. 5. Regression coefficients $R_1(S)$, the thicker line, $R_S(1)$ - the thinner curve

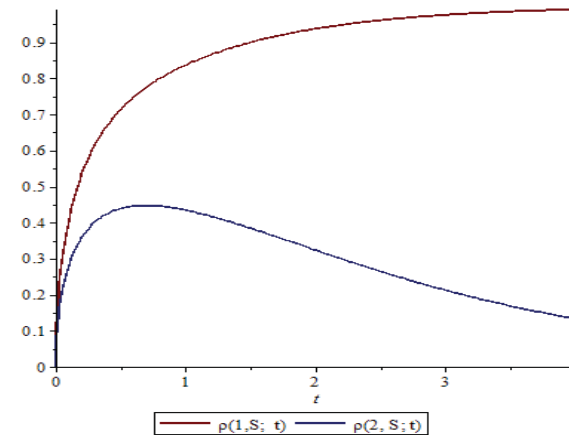


Fig. 5. Correlation coefficients $\rho_1(S)$, the thicker line, and $\rho_S(1)$ (the thinner)

CONCLUSIONS

- We discussed measures of dependence between two random events.
- These measures are equivalent, and exhibit natural properties.
- Their numerical values may serve as indication for the magnitude of dependence between random events.
- These measures provide simple ways to detect independence, coincidence, degree of dependence.
- If either measure of dependence is known, it allows better prediction of the chance for occurrence of one event, given that the other one occurs.

CONCLUSIONS

- We extend the use of these measures from events to local dependence between random variables
- Our study of the local dependence is on rectangles where interval values of the random variables meet. It exhibits different behavior than the global dependence.
- The local dependences are universally valid and can be continued for higher dimensions.
- Numerical illustrations (for politics and reliability systems, and non-numeric social study) confirm our expectations.
- We show that **local dependence can be essentially different on different areas in the field.**

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