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TRANSFORMATIONS OF INDEX SET FOR SKOROKHOD INTEGRAL WITH RESPECT TO GAUSSIAN PROCESSES

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We consider a Gaussian process \( \{X_t, t \in T\} \) with an arbitrary index set \( T \) and study consequences of transformations of the index set on the Skorokhod integral and Skorokhod derivative with respect to \( X \). The results applied to Skorokhod SDEs of diffusion type provide uniqueness of the solution for the time-reversed equation and, to Ogawa line integral, give an analogue of the fundamental theorem of calculus.

**Key words:** Skorokhod Integral, Anticipative Stochastic Calculus.

**AMS subject classifications:** 60H05, 60H10.

1. Introduction

The purpose of this article is to prove that, in a general case of Gaussian processes and under mild assumptions, transformations of a parameter set do not change the Skorokhod integral and Skorokhod derivative, and to indicate some applications of this fact.

Let \( T \) be any set, \( C \) a covariance on \( T \) and \( H(C) = H \) the reproducing kernel Hilbert space (RKHS) on \( C \) (note that \( H \) may not be separable). With covariance \( C \), we associate a Gaussian process \( \{X_t, t \in T\} \) defined on \( (\Omega, \mathcal{F}, P) \), where \( \mathcal{F} = \sigma\{X_t, t \in T\} \). For the details of the constructions above, see [3]. Let \( H \otimes p \) be the \( p \)-fold tensor product of \( H \). The \( p \)-Multiple Wiener Integral (MWI) \( I_p : H \otimes p \rightarrow L_2(\Omega, \mathcal{F}, P) \) was defined in [6] (see also [5]) as a linear mapping satisfying the following properties. Here \( \tilde{f} \) is the symmetrization of \( f \).

\begin{align*}
& a) \quad EI_p(f) = 0, \\
& b) \quad EI_p(f)I_q(g) = \left\{ \begin{array}{ll}
0 & \text{if } p \neq q, \\
\hat{p}(\tilde{f}, \tilde{g})_H \otimes p & \text{if } p = q,
\end{array} \right. \quad \text{for } f \in H \otimes p, \ g \in H \otimes q.
\end{align*}

\[ c) \quad I_{p+1}(gh) = I_p(g)I_1(h) - \sum_{k=1}^{p} I_{p-1}(g \otimes h), \ \text{for } g \in H \otimes p, \ h \in H. \]

Above, \( (g \otimes h) (t_1, \ldots, t_k, t_{k+1}, \ldots, t_p) = (g(t_1, \ldots, t_k, \cdot, t_{k+1}, \ldots, t_p), h(\cdot))_H. \)
We note that $I_p(f) = I_p(f)$ and hence $I_p(H \otimes p) = I_p(H \otimes p)$ where $H \otimes p$ is the $p$-fold symmetric tensor product.

Let $u: \Omega \to H$ be a Bochner measurable function with $\|u\|_H \leq L_2(\Omega, \mathbb{F}, P)$. Using Wiener chaos decomposition, $L_2(\Omega, \mathbb{F}, P) = \sum_{p=0}^{\infty} \otimes p I_p(H \otimes p)$, we have a unique representation $u_t(\omega) = \sum_{p=1}^{\infty} \otimes p I_p(f_p(\cdot, t))$, with $f_p(\cdot, \cdot) \in H \otimes p + 1$ and $f_p(\cdot, t) \in H \otimes p$. The Skorokhod derivative and integral of $u$ with respect to Gaussian processes are defined in [6] (for Skorokhod’s original definition, see [12]). The Skorokhod derivative $\{D^8 u_t, s \in T\}$ of $u_t$ for a fixed $t$ is an $L_2(n, H)$-valued random variable, $D^8 u_t = \sum_{p=1}^{\infty} p I_p(f_p(t_1, \ldots, t_p-1, s, t))$. The Skorokhod derivative exists iff $\mathbb{E}\|D^8 u_t\|_p^2 = \sum_{p=1}^{\infty} p! \|f_p(\cdot, t)\|_{H \otimes p}^2 < \infty$ and $\{D^8 u_t \in L_2(\Omega, H \otimes 2), s, t \in T\}$, with $H \otimes 2$ identified with the space of Hilbert-Schmidt operators on $H$, iff $\mathbb{E}\|D^8 u_t\|_{H \otimes 2}^2 = \sum_{p=1}^{\infty} p! \|f_p\|_{H \otimes (p+1)}^2 < \infty$.

The Skorokhod integral of $u$ is an $L_2(0)$-valued random variable, $I^s(u) = \sum_{p=0}^{\infty} I_p(f_p(\cdot, *))$. We note that $u$ is integrable iff $\mathbb{E}\|I^s(u)\|^2 = \sum_{p=0}^{\infty} (p+1)! \|f_p(\cdot, *)\|_{H \otimes p+1}^2 < \infty$.

**Example 1:** Skorokhod derivative and integral for Brownian motion. In the case of standard Brownian motion, the MWI $I_p$ and consequently, the Skorokhod derivative and integral defined above, coincide with the MWI $I^t$, the Malliavin derivative $D^t$ and the Skorokhod integral $I^t$ defined in [7]. With $V: L_2([0, 1]) \to H$ defined by: $V f = \int_0^1 f(s) ds$, $I^t(V f) = I_p(V \otimes p f)$, $I^t(V(u)) = I^t(u)$ and $D^t(V(u)(t)) = D^t u_t$ for $f_p \in L_2([0, 1]^p)$ and $u \in L_2(\Omega, L_2([0, 1]))$. The first two equalities hold in $L_2(\Omega)$ and the third holds in $L_2(\Omega, H)$ for a fixed $t$.

If $u$ is adapted to the natural (resp. future) filtration of Brownian motion, $\mathbb{F}_t = \sigma(B_s, s \leq t)$ ($\mathbb{F}^t = \sigma(B_1 - B_s, t \leq s \leq 1)$), then the Skorokhod and Itô (backward Itô) integrals coincide (see [7]).

### 2. Skorokhod Integral Under Transformation of a Parameter Set

For a Gaussian process $\{X_t, t \in T\}$, let $H(X) = cl(span\{X_t, t \in T\})$, the closure being taken in $L_2(\Omega, \mathbb{F}, P)$. With a transformation $R: S \to T$ we associate a Gaussian process $X^R = \{X^R(s) \mid s \in S\}$ and we call $R$ nondegenerate if it is onto and if $H(X^R) = H(X)$. Our main result on transformations of the Skorokhod derivative and integral is the following:

**Theorem 1:** Let $\{X_t\}_{t \in T}$ be a Gaussian process and $R: S \to T$ be a nondegenerate transformation. Denote by $I^s_X$ and $I^s_{X^R}$ the Skorokhod integrals with respect to $X$ and $X^R$, respectively. Then:

1. $f_p \mapsto \widehat{f}_p = f(R(s_1), \ldots, R(s_p))$ is an isometry from $H(C_X) \otimes p$ onto $H(C_X^R) \otimes p$.

2. If $u \in \mathbb{F}(I^s_X)$ then $u^R = \{u^R(s) \mid s \in S\} \in \mathbb{F}(I^s_{X^R})$ and $I^s_{X^R}(u) = I^s_{X^R}(u^R)$.
Moreover, denote by $D^X$ and $D^{X_R}$ the Skorokhod derivatives with respect to $X$ and $X_R$, respectively.

3) If for $t \in T$, $u_t \in \mathcal{B}(D^X)$, then $u_s^R \in \mathcal{B}(D^{X_R})$ for $s \in R^{-1}(t)$ and $D^{X_R}_{s'} u^R_s = D^X_{R(s')} u^R_s$ $P$-a.e., for $s, s' \in S$. The equality is in $H(C_{X_R})$, with $s' \in S$ as the variable.

Also, $D^X_t u_t \in H(C_X) \otimes_2$, $(t, t' \in T)$ implies $D^{X_R}_{s'} u^R_s \in H(C_{X_R}) \otimes_2$, $(s, s' \in S)$, and equality of norms $\| D^X_t u_t \|_{L_2(\Omega, H(C_X) \otimes_2)} = \| D^{X_R}_{s'} u^R_s \|_{L_2(\Omega, H(C_{X_R}) \otimes_2)}$.

4) If $v \in L_2(\Omega, H(C_{X_R}))$ then $v = u^R$ for some $u \in L_2(\Omega, H(C_X))$ and $\| v \|_{L_2} = \| u \|_{L_2^*}$.

Moreover, $u \in \mathcal{B}(I^R_X)$ implies $u \in \mathcal{B}(I^R_X)$ and $v \in \mathcal{B}(D^{X_R})$ implies $u_{R(s)} \in \mathcal{B}(D^X)$ with $D^{X_R}_{s'} u^R_s = D^X_{R(s')} u^R_s$ for $s, s' \in S$.

If $D^{X_R}_{s'} v_s \in H(C_{X_R}) \otimes_2$, $(s, s' \in S)$, then $D^X_t u_t \in H(C_X) \otimes_2$,

Proof: 1) Let us denote $f^R(s_1, \ldots, s_n) = f(R(s_1), \ldots, R(s_n))$ for $(s_1, \ldots, s_n) \in S^p$, (thus $f^R_p(s_1, \ldots, s_p, s) = f_p(R(s_1), \ldots, R(s_p), R(s))$, $(s_1, \ldots, s_p, s) \in S^{p+1}$). Let $f(t) \in H(C_X)$, then $f(t) = E(X_t I^X_1(f))$, with $I^X_1(f) \in H(X)$ and, for any $s \in S$,

$$f^R(s) = f(R(s)) = E(X_{R(s)} I^X_1(f)) = E(X^R_I f)$

($I^X_p$ or $I^{X_R}_p$ denotes the $p^{th}$ order Wiener integral with respect to either $X$ or $X^R$).

By definition and uniqueness of representation, $f^R \in H(C_{X_R})$ and $I^X_1(f^R) = I^X_1(f)$. Also, if $g \in H(C_{X_R})$ then, for $s \in S$, $g(s) = E(X_{R(s)} I^X_1(g))$. But, $I^X_1 (g) \in H(X)$, thus $f(t) = E(X_t I^X_1(g))$ defines an element of $H(C_X)$, with $g(s) = f(R(s))$, $s \in S$ and $\| g \|_{H(C_{X_R})} = \| I^X_1 g \|_{L_2(\Omega, H(C_{X_R}), P)} = \| f \|_{H(C_X)}$, proving (1).

2) - 3) Let us first show that $I^X_p(f_p) = I^X_p(f^R_p)$, $p = 0, 1, \ldots$

The above is clear for $p = 0$ and $p = 1$. Let $f_p \in H(C_X) \otimes_p$, $f(t_1, t_2, \ldots, t_p) = \sum \alpha_1, \alpha_2, \ldots, \alpha_p a_1^p \alpha_1 \alpha_2 \ldots e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) \ldots e_{\alpha_p}(t_p)$, with $\sum \alpha_1, \alpha_2, \ldots, \alpha_p < \infty$ and $\{ e_{\alpha} : \alpha = 1, 2, \ldots \}$ an ONB in $H(C_X)$. For $f_p = e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) \ldots e_{\alpha_p}(t_p)$ we have $[(f_p \otimes g_1)^X]R(s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_p) = (f^R_p \otimes g^R_1)^R(s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_p)$, where the superscripts $X$ and $X^R$ indicate that the operation $ \otimes $ is taken either with respect to the process $X$ or $X^R$. Thus, $I^X_p((f_p \otimes g_1)^X) = I^X_p ((f^R_p \otimes g^R_1)^R) = I^X_p ((f^R_p \otimes g^R_1)^X) R$, which allows us to use the inductive relation (c) for MWI to complete the proof. For $f_p \in H(C_X)$ arbitrary,
we have

\[ I^X_p(f_p) = \lim_{n_1, \ldots, n_p \to \infty} I^X_p \left( \sum_{\alpha_1=1}^{n_1} \cdots \sum_{\alpha_p=1}^{n_p} a_{\alpha_1} \cdots a_p e_{\alpha_1} \cdots e_{\alpha_p} \right) \]

\[ = \lim_{n_1, \ldots, n_p \to \infty} I^X_p \left( \sum_{\alpha_1=1}^{n_1} \cdots \sum_{\alpha_p=1}^{n_p} a_{\alpha_1} \cdots a_p e_{\alpha_1} \cdots e_{\alpha_p}^R \right) \]

\[ = I^X_p \left( \lim_{n_1, \ldots, n_p \to \infty} \left( \sum_{\alpha_1=1}^{n_1} \cdots \sum_{\alpha_p=1}^{n_p} a_{\alpha_1} \cdots a_p e_{\alpha_1} \cdots e_{\alpha_p}^R \right) \right) = I^X_p (f_p^R). \]

Now if \( u \in \mathcal{D}(I^X_p) \) and \( u_t = \sum_{p=0}^{\infty} I^X_p(f_p(t_1, \ldots, t_p, t)) \) then, for \( s \in S \),

\[ u_{R(s)} = \sum_{p=0}^{\infty} I^X_p(f_p(\cdot, R(s))) = \sum_{p=0}^{\infty} I^X_p (f_p^R(\cdot, s)) \]

and 2) and 3) follow.

4) Let \( v \in L_2(\Omega, H(C_X \otimes \mathbb{R})); \) then for \( s \in S \), using 1),

\[ v_s = \sum_{p=0}^{\infty} I^X_p(g_p(\cdot, s)) = \sum_{p=0}^{\infty} I^X_p (f_p^R(\cdot, s)), \]

because for any \( g \in H(C_X \otimes \mathbb{R})^{(p+1)} \) there exists \( f \in H(C_X \otimes \mathbb{R})^{(p+1)} \) with \( g = f^R \). Hence, for \( s \in S \), \( v_s = \sum_{p=0}^{\infty} I^X_p (f_p^R(\cdot, s)) = \sum_{p=0}^{\infty} I^X_p (f_p^R(\cdot, R(s))) \).

According to 1), \( u_t = \sum_{p=0}^{\infty} I^X_p (f_p^R(\cdot, t)) \in L_2(\Omega, H(C_X)) \) and equality of norms claimed in 4) is satisfied. The last part of assertion 4) follows from 1), 2) and 3) since failure to satisfy any stated condition by \( u \) implies violation of this condition by \( v \).

\[ \square \]

**Example 2:** Transformations of parameter set and Skorokhod integral.

1) Brownian motion and time reversal. Let \( \{u_t, t \in [0, 1]\} \) be an \( L_2(\Omega, L_2[0, 1]) \)-valued process adapted to the natural filtration \( (\mathcal{F}_t)_{t \in [0, 1]} \) of Brownian motion. Note that \( \{\widetilde{B}_t = B_1 - B_{1-t}, t \in [0, 1]\} \) is also a Brownian motion and \( \{\overline{u}_t = u_{1-t}, t \in [0, 1]\} \) is adapted to filtration \( \mathcal{G}^t = \sigma(\widetilde{B}_1, \widetilde{B}_s, t \leq s \leq 1) \). Denote \( \widehat{B}_t = B_{1-t} \).

We have

\[ \int_0^1 u_t dB_t = I^s_B \left( \int_0^1 u_t \, dr \right) = I^s_B \left( \int_0^{1-t} ur \, dr \right). \]

By the same method as in the proof of Theorem 1 we can show that

\[ I^s_B ((\int_0 u_t \, dr) \sim) = I^s_B (\int_0 u_t \, dr) \] with \( (\int_0 u_t \, dr) \sim = \int_0 u_t \, dr - \int_0^{1-t} u_t \, dr \). Hence we get

\[ \int_0^1 u_t dB_t = I^s_B \left( \int_0^{1-t} u_t \, dr \right) = I^s_B (\overline{u}) = \int_0^1 \overline{u}_t \, d\widehat{B}_t \]

where \( \sim * \) denotes the backward Itô integral. We have just obtained the relation
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\[ T_{\tau}(u) = I_{\tau}^\infty(u) \] given in [8]. Note also that \( \tilde{B}_t \) is not a Brownian motion and equation (1) is reversed pathwise in \( H \). In the case of Brownian motion, we also have

\[ I_{\tilde{B}}^\infty\left( \int_0^1 \cdot u_s ds \right) = I_{\tilde{B}}^\infty\left( \int_0^1 \cdot u_s ds \right) \sim. \]

Indeed,

\[ I_{\tilde{B}}^\infty\left( \int_0^1 \cdot u_s ds \right) = I_{\tilde{B}}^\infty\left( \int_0^1 \cdot u_s ds \right) = I_{\tilde{B}}^\infty\left( \int_0^1 \cdot u_s ds \right) = I_{\tilde{B}}^\infty\left( \int_0^1 \cdot u_s ds \right). \]

2. Ogawa Line Integral. We recall the definition of the Ogawa integral ([4, 9]) with respect to a Gaussian process \( \{X_t, t \in [0,1]\} \) with the RKHS \( H \). Let \( u: \Omega \rightarrow H \) be an \( H \)-valued Bochner measurable function. Then, on a set of \( P \)-measure one, \( u(\omega) \) takes values in a separable subspace of \( H \). Let \( \{e_n, n \in \mathbb{N}\} \) be an ONB of this subspace. The (universal) Ogawa integral of \( u \) is defined as follows:

\[ \delta(u) = \sum_{n=1}^{\infty} (u, e_n)_H I_1(e_n) \text{ (limit in probability)} \]

if it exists with respect to all ONBs and is independent of the choice of basis.

The relation between Skorokhod and Ogawa integrals is explained in [4].

Let \( \gamma:S \rightarrow T \) be a bijective parametrization. Let \( Y_s = X_{\gamma(s)} \). Then

(i) \( C_X(\gamma(s_1), \gamma(s_2)) = C_Y(s_1, s_2) \);

(ii) \( H(C_X) \) and \( H(C_Y) \) are isometric under the mapping \( f \mapsto f \circ \gamma \);

(iii) \( I_{\gamma}^X(f) = I_{\gamma}^Y(f \circ \gamma) \) for \( f \in H(C_X) \).

Thus, \( \delta_X(u) = \delta_Y(v) \) for \( v_s = u_{\gamma(s)} \) provided either of the integrals exists.

Consider Brownian sheet \( \{W(x,t), (x,t) \in [0,1]^2\} \). Assume that \( \Gamma \subset [0,1]^2 \) is a curve parametrized by a function \( \gamma:[a,b] \rightarrow \Gamma, \ 0 \leq a \leq b \leq 1 \). We define the Ogawa line integral, \( \Gamma - \delta \), over \( \Gamma \) with respect to \( \{W(x,t), (x,t) \in \Gamma\} \) using \( \Gamma \) as the parameter set. In addition, let \( \gamma(s) = (\gamma_1(s), \gamma_2(s)) \) with both coordinates nondecreasing and such that the map \( \gamma^{-1}(\gamma_1(r), \gamma_2(r)) = (\gamma_1(r) \gamma_2(r)) \) is bijective from \( \Gamma \) to \( S = [\gamma_1(a), \gamma_2(a), \gamma_1(b), \gamma_2(b)] \). Then \( \gamma:S \rightarrow \Gamma \) is a bijective parametrization and the process \( B_s = W_{\gamma}(s) \) is a Brownian motion. Hence,

\[ \Gamma - \delta_W(u) = \delta_B(v) = \int_S (V^{-1}v)(s) \circ dB_s, \]

where \( v_s = u_{\gamma(s)} \); \( V \) is the isometry from Example 1, and the last integral is in the sense of Fisk and Stratonovich and is assumed to exist. In particular, if \( u(x,t) = f(W(x,t)) \) and \( f \in C^2 \), then

\[ \Gamma - \delta_W(V \otimes^2 f'(W)) = \int_S f'(B_s) \circ dB_s = f(W(\gamma_1(b), \gamma_2(b))) - f(W(\gamma_1(a), \gamma_2(a))). \]

Thus, in this case, the Ogawa line integral satisfies the fundamental theorem of calculus. We conjecture that a counterpart of Green’s formula for the Ogawa integral holds (see [2] for initial exposition and [11] for some recent results).
Example 3: Skorokhod-type stochastic differential equations. The following class of Skorokhod SDEs was considered by Buckdahn in [1], where, under smoothness assumptions, the author proved existence and uniqueness results

\[ Z_t = \eta + \int_0^t b(Z(s))ds + \int_0^t \sigma(Z(s))1_{[0,t]}(s)B_s, \quad 0 \leq t \leq 1. \] (2)

The initial condition \( \eta \) needs to be bounded. However, this restriction vanishes if equation (2) is reversed.

Lemma 1: Let \( \{u_t\} \in [0,1] \) be such that \( u_{[0,t]}(s) \in \mathcal{D}(I_B) \) \( \forall t \in [0,1] \). Then for the time reversed process \( \tilde{u}_t = u_{1-t} \), we have \( \tilde{u}_{[0,t]}(s) \in \mathcal{D}(I_B) \) \( \forall t \in [0,1] \) and if we denote \( X_t = \int_0^t 1_{[0,t]}(s)u_s \), then

\[ X_{1-t} - X_1 = -\int_0^t 1_{[0,t]}(s)\tilde{u}_s. \]

Using time reversal and Lemma 1, Buckdahn’s result can be extended to time reversed SDEs with the initial condition being a terminal value of the solution of the original equation.

Theorem 2: Assume that coefficients \( b \) and \( \sigma \) of a Skorokhod SDE (2) satisfy assumptions for existence and uniqueness of the solution. If \( \{Z_t\} \in [0,1] \) is the solution of Equation (2), then the time reversed process \( \tilde{Z}_t = Z_{1-t} \) is the unique solution in \( L_1((0,1] \times \Omega) \) of the time reversed equation

\[ X_t = \tilde{Z}_0 + \int_0^t -\tilde{b}(X_s)ds + \int_0^t \tilde{\sigma}(X_s), \]

where \( \tilde{b}(X_t) = b(X_{1-t}), \tilde{\sigma}(X_t) = \sigma(X_{1-t}), \) and \( \tilde{B}_t = B_1 - B_{1-t}. \)

The above theorem gives a partial answer to a question in [8], Proposition 5.2.

The technique of time reversal has been used in [10] to solve a problem regarding anticipative stochastic models in finance.

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